Burning cars in a parking

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Abstract

Knuth's parking scheme is a model in computer science for hashing with linear probing. One may imagine a circular parking with n sites; cars arrive at each site with unit rate. When a car arrives at a vacant site, it parks there; otherwise it turns clockwise and parks at the first vacant site which is found. We incorporate fires to this model by throwing Molotov cocktails on each site at a smaller rate $n^{-\alpha}$ where $0 < \alpha < 1$ is a fixed parameter. When a car is hit by a Molotov cocktail, it burns and the fire propagates to the entire occupied interval which turns vacant. We show that with high probability when $n \to \infty$, the parking becomes saturated at a time close to 1 (i.e. as in the absence of fire) for $\alpha > 2/3$, whereas for $\alpha < 2/3$, the mean occupation approaches 1 at time 1 but then quickly drops to 0 before the parking is ever saturated. Our study relies on asymptotics for the occupation of the parking without fires in certain regimes which may be of independent interest.

Key words: Knuth parking scheme, forest fire, phase transition.

1 Introduction

The purpose of this work is to point at a phase transition for a random evolution which combines the dynamics of two different models in statistical physics and computer science, namely forest fires and parking schemes. Its motivation partly stems from an interesting paper by Ráth and Tóth [16] in which the authors introduce forest fires in the Erdős-Rényi random graph model. It is well-known that the random graph model is closely related to the multiplicative coalescent [1], and the incorporation of fires causes random shattering of large components. On the other hand, it is also known that the parking scheme bears close connexions to the additive coalescent [11, 6] and it seems therefore natural to investigate the effects of random shattering in this framework.

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Forest fires have been introduced by Drossel and Schwabl [12], see also the review [17]; they are prototypes of systems displaying self-organized criticality. Typically, imagine a lattice where each site is either vacant or occupied by a tree; the connected components of occupied sites are viewed as forests. Each vacant site becomes occupied at some fixed rate, independently of the others. One may think of seeds being sown uniformly on the lattice; a tree grows each time a seed falls on a vacant site and seeds falling on a site that is already occupied are discarded. Furthermore, each tree can also be hit by a lightning at some rate. Then the tree burns and the fire propagates to the entire forest, i.e. any site that can be connected by a path consisting only of occupied sites to the site hit by a lightning becomes instantaneously vacant. So, roughly speaking, forests may grow by addition of trees on their boundaries or coalescence when a vacant site separating two forests turns occupied, and disappear when hit by a lightning. The evolution of the system thus results from two opposite trends growth/destruction which should be viewed as the source of self-organization.

Although forest fires have raised a considerable interest, notably in the physical literature, mathematical papers on these models are still rather scarce; see e.g. [2, 3, 7, 9, 13, 14]. The recent paper [8] by Bressaud and Fournier deals with the more general situation where seeds and lightnings occur according to stationary renewal processes, and contains in particular a detailed and well documented introduction to the subject. The work by Ráth and Tóth [16] which we already mentioned is somewhat special because the underlying lattice is the complete graph, and basically this circumvents geometrical difficulties. The study of asymptotics when the rate of lightning tends to 0 is especially challenging since, informally, a low rate of lightning should enable forests to grow larger although the larger a forest grows, the more inflammable it becomes.

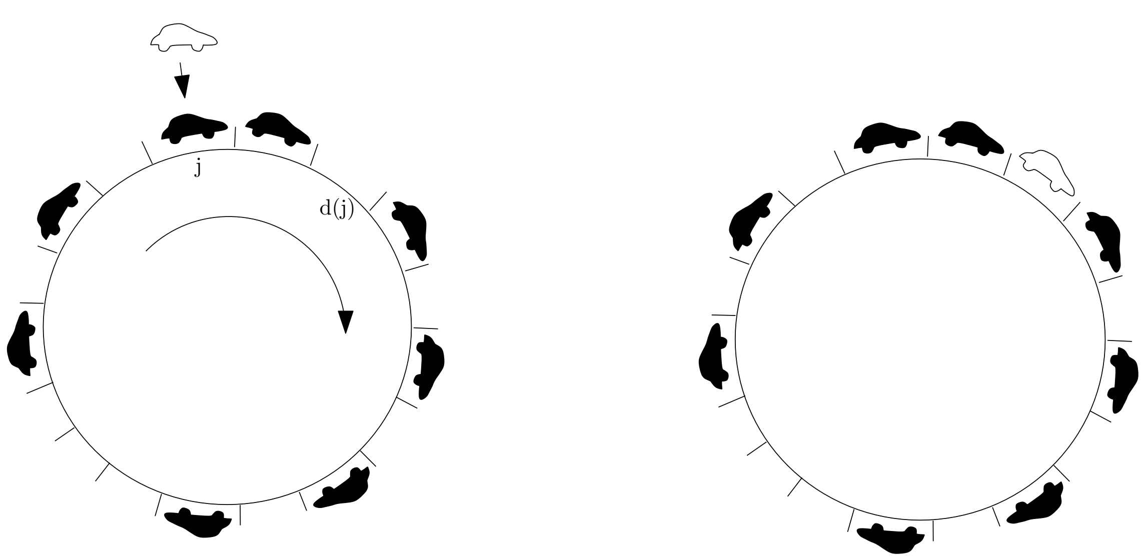

Figure 1 : *A car arrives at an occupied site j and parks at the first vacant site $d(j)$.*

On the other hand, Knuth's parking is a simple model in computer science for hashing with linear probing; we refer the French readers to the very nice survey by Chassaing and Flajolet [10]. Think of $\mathbb{Z}/n\mathbb{Z}$ as a circular parking where initially all sites are vacant. Cars arrive one after the other, uniformly at random. If a car arrives at a site $j \in \mathbb{Z}/n\mathbb{Z}$ which is vacant, then the car parks at j and this site becomes occupied. Otherwise the car tries to park successively at sites $j+1, j+2, \ldots$ until it finally finds a vacant site, see Figure 1 above.

In this work, we incorporate fires to Knuth's parking model. We may imagine that Molotov cocktails are thrown on the parking, and when a Molotov cocktail falls on an occupied site, the car parked there burns, the fire propagates to the neighboring cars, and the entire connected component of occupied sites becomes instantaneously vacant. See Figure 2 below.

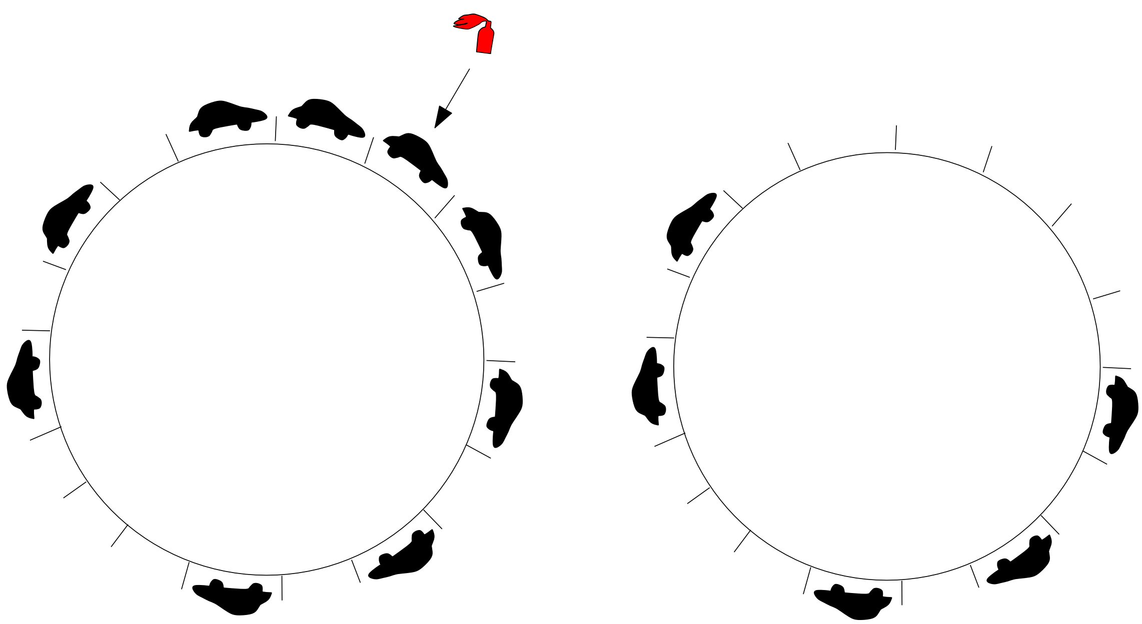

Figure 2 : *A Molotov cocktail is thrown on an occupied interval which then becomes vacant.*

A fundamental difference with the forest fire model on $\mathbb{Z}/n\mathbb{Z}$ is that a car may occupy a site different from that at which it arrived, whereas trees only grow on sites on which a seed falls. Empty blocks have clearly an important role in the dynamics as, roughly speaking, they forms barriers that prevent the propagation of fires. In particular, this ensures the independence of the evolutions of regions separated by such barriers. In the case of forest fires, a barrier remains effective until a seed has fallen on every single site of that block, which takes typically a long time when the block is large. However, the occupation of vacant intervals in a parking may occur much quicker when, loosely speaking, the portion of the parking at the left of this interval is already densely occupied. Thus the parking model involves a stronger spacial dependency which makes the study even more tedious.

In this work, we assume that cars arrive on each site of $\mathbb{Z}/n\mathbb{Z}$ at a unit rate whereas Molotov cocktails hit each site at rate $n^{-\alpha}$ for some parameter $\alpha \in (0,1)$, independently of car arrivals. We are interested in saturation, i.e. when the parking is entirely occupied; note that saturation times are renewal epochs for the dynamics. Observe that when no Molotov cocktails are thrown, the parking becomes saturated when n cars have arrived, and thus at a time close to 1. We are interested in the question of whether Molotov cocktails have a significant impact on the saturation of the parking. Our main result shows that $\alpha = 2/3$ is the critical exponent. More precisely, if $\alpha > 2/3$, then with high probability[1] the parking becomes saturated at a time close to 1, while if $\alpha < 2/3$, even though the mean occupation of the parking approaches 1 as time tends to $1-$, it then quickly drops to 0 before the parking is completely saturated.

Let us explain intuitively this phase transition. When $\alpha > 2/3$, Molotov cocktails are rather scarce and one can show that the total number of cars which have been burnt until the arrival of the m-th car with $m \approx n - n^{\beta}$ for some $2/3 < \beta < \alpha$ can be bounded from above by n^{β}. During approximatively $2n^{\beta-1}$ units of times, $2n^{\beta}$ new cars arrive on the parking while the probability that a Molotov cocktail is thrown is only of order $n^{1-\alpha} \times 2n^{\beta-1} \ll 1$. This suffices to saturate the parking. The case $\alpha < 2/3$ is more complex. Roughly speaking, the crucial step consists in establishing the existence of some $\beta > \alpha$ and of a small time window before 1 during which mesoscopic occupied intervals of size of order n^{β} are formed. In particular, the mean occupation of the parking is close to 1 at such times. Since $n^{\beta-\alpha} \gg 1$, with high probability such mesoscopic intervals are entirely destroyed by a Molotov cocktail shortly after they have been formed, which causes the mean occupation to quickly drop to 0. More precisely, during this short time interval, destruction of cars by fires overpasses significantly the arrivals of new cars, and thus prevents the saturation of the parking.

An obvious difficulty in proving rigorously these statements, and in particular in finding the critical parameter, lies in the fact that most of the relevant phenomena occur during a very short time when mesoscopic occupied intervals are formed before being quickly destroyed by fires. Our approach to investigate their formation consists in establishing that for certain times close to 1, fires still have essentially a negligible impact on the occupation of the parking. In particular, we are led to analyze asymptotics of the occupation process for parking schemes without fires in intermediate regimes which do not seem to have been considered previously in the literature.

The plan of the rest of this work is as follows. The next section is devoted to preliminary

[1] We say that an event depending on n holds *with high probability* if its probability tends to 1 as $n \to \infty$. Similarly, we say that a random variable depending on n is close to c (respectively, is small) with high probability if it converges in probability to c (respectively, to 0) as $n \to \infty$.

observations in the deterministic setting. In particular we shall see that the occupation of the parking can be described in terms of a quasi-periodical path constructed from the processes of cars and Molotov cocktails. This will enables us to relate different parkings based on comparable dynamics. Section 3 deals with key asymptotic estimates for parkings without fires. Specifically, we shall establish limit theorems for the empirical measure of the sizes of occupied intervals in a parking of size n when $m < n$ cars have arrived, in certain regimes when $n, m \to \infty$. Our main result is stated and proved in Section 4. The argument relies on an intermediate claim which, roughly speaking, states that the impact of fires is essentially negligible until the arrival of the m-th car provided that m is not too close to n (of course it is crucial to be able to let m be as large as possible). In other words, asymptotics for the occupation of the parking are then the same as for the dynamics without fires which have been studied in the preceding section. The proof of this technical result is presented in Section 5.

2 Preliminaries in the deterministic setting

2.1 Analysis of the model in terms of paths

For $n \in \mathbb{N}$ fixed, a configuration for a parking with n sites is a map $\omega : \mathbb{Z}/n\mathbb{Z} \to \{0, 1\}$; the site j is vacant if $\omega(j) = 0$, and is occupied otherwise. We write $\mathbf{0}$ for the configuration when the parking is totally vacant (i.e. for the map $\omega(j) \equiv 0$) and $\mathbf{1}$ for the configuration corresponding to saturation. The support of a configuration corresponds to the set of occupied sites; its connected components are called occupied intervals, or sometimes arcs.

We represent car arrivals by a point process $(C_t, t \geq 0)$ on $\mathbb{Z}/n\mathbb{Z}$, where $C_t = j$ if a car arrives at time t on the site j and $C_t = \varnothing$ if no car arrives at time t. Similarly, Molotov cocktails form another point process $(M_t, t \geq 0)$ on $\mathbb{Z}/n\mathbb{Z}$; we implicitly assume that times when Molotov cocktails are thrown never coincide with the arrival time of a car. The occupation of the parking is given by a process $(\Theta_t, t \geq 0)$ with values in the space of configurations which is constructed from these two point processes as follows. We assume that the parking is completely vacant at the initial time, viz. $\Theta_0 = \mathbf{0}$. The occupation remains unchanged on every time-interval during which no car arrives and no Molotov cocktail is thrown. Suppose first that a car arrives at time $t > 0$ on the site j, i.e. $C_t = j$. If the parking is already saturated, i.e. $\Theta_{t-} = \mathbf{1}$, then we decide that $\Theta_t = \mathbf{1}$. Otherwise we denote [2] by $D_{t-}(j)$ the closest vacant site of Θ_{t-} to the right of j for the cyclic order (if j is vacant at time t, then $D_{t-}(j) = j$), and we set

$$\Theta_t = \Theta_{t-} + \mathbf{1}_{\{D_{t-}(j)\}} .$$

[2] The notation D and G refer to *droite* (right) and *gauche* (left) in French; they are commonly used when dealing with extremities of components of random subsets of the real line

On the other hand, suppose that a Molotov cocktail is thrown at time t on the site j, i.e. $M_t = j$. If the parking is saturated, i.e. $\Theta_{t-} = \mathbf{1}$, then it instantaneously becomes vacant, i.e. $\Theta_t = \mathbf{0}$. This should be viewed as a renewal time for the occupation process. If j is already vacant, i.e. $\Theta_{t-}(j) = 0$, then $\Theta_t = \Theta_{t-}$. Finally, if j is occupied but the parking is not saturated, then we write $G_{t-}(j)$ and $D_{t-}(j)$ for the first vacant sites at the left and at the right of j and set

$$\Theta_t = \Theta_{t-} - \mathbf{1}_{]G_{t-}(j), D_{t-}(j)[}\,.$$

A key tool for the analysis is that the occupation of the parking at some time t can be conveniently described in terms of a quasi-periodic path. This is a simple observation for parking without fires (see for instance Chassaing and Louchard [11]), which is easily extended in the present setting. Specifically, denote for every $j \in \mathbb{Z}$ by $\xi_t(j)$ the number of cars that have arrived before time t at the site $j\,[\mathrm{mod}\, n]$ and are parked at time t, either at j or further away on the parking. We stress that we do not take into account cars which have been burnt before time t nor those which have arrived at a time when the parking was already saturated.

Define a path $S_t : \mathbb{Z} \to \mathbb{Z}$ by

$$S_t(0) = 0 \quad \text{and} \quad S_t(j) - S_t(j-1) = \xi_t(j) - 1\,, \qquad \forall j \in \mathbb{Z}\,. \tag{1}$$

Clearly S_t is quasi-periodic, in the sense that $S_t(j+n) = S_t(j) + S_t(n)$ for any $j \in \mathbb{Z}$. Note the number of vacant sites on the parking is given by $S_t(n)^-$, the negative part of $S_t(n)$; in particular the parking is saturated if and only if $S_t(n) = 0$. The running minimum

$$\underline{S}_t(k) = \min_{j \le k} S_t(j)\,, \qquad k \in \mathbb{Z}$$

is constant when $S_t(n) = 0$, and otherwise is a non-degenerate quasi-periodic path. We now make the key observation that a site $j\,[\mathrm{mod}\, n]$ of the parking is vacant at time t if and only if the path S_t reaches a new minimum at j.

Lemma 1 *In the notation above, we have for any $j \in \mathbb{Z}$*

$$\Theta_t(j\,[\mathrm{mod}\, n]) = 0 \iff S_t(j) < \underline{S}_t(j-1).$$

As a consequence, provided that the parking is not saturated (i.e. if $S_t(n) < 0$), the interval of occupied sites at time t that contains 0 is given by $]G_t, D_t[$ with

$$G_t = \max\{j \le 0 : \underline{S}_t(j-1) > \underline{S}_t(0)\} \text{ and } D_t = \min\{j \ge 0 : \underline{S}_t(j) < \underline{S}_t(-1)\}\,.$$

Note that $G_t = D_t = 0$ (and thus $]G_t, D_t[= \varnothing$) if and only if the site 0 is vacant at time t.

Proof: We shall check the first statement by induction. Denote by $0 < t_1 < t_2 < \dots$ the sequence of times at which either a car arrives or a Molotov cocktail is thrown. The claim is obvious for $t = t_1$; assume that it holds for $t = t_m$ for some $m \in \mathbb{N}$.

Consider first the case when a Molotov cocktail is thrown at time t_{m+1}. Plainly our claim still holds at time t_{m+1} when either the parking is saturated at time t_m or the Molotov cocktails falls on a vacant site. So assume that it falls on some occupied interval $I \neq \mathbb{Z}/n\mathbb{Z}$ and denote by g and d the first vacant sites at the left and the right of I, i.e. $I =]g, d[$. By construction, we have $\xi_{t_{m+1}}(j) = 0$ if $j\,[\text{mod } n] \in I$ and $\xi_{t_{m+1}}(j) = \xi_{t_m}(j)$ otherwise. Thus the path $S_{t_{m+1}}$ is derived from S_{t_m} by replacing all the steps on $I\,[\text{mod } n]$ by -1 and leaving the others unchanged. As the sites g and d are vacant at time t_m, our induction assumption ensures that

$$S_{t_m}(g) < \underline{S}_{t_m}(g-1) \quad \text{and} \quad S_{t_m}(d) < \underline{S}_{t_m}(d-1)\,,$$

from which we readily conclude that the claim still holds at time t_{m+1}.

Next, consider the case when a car arrives at time t_{m+1}, say on the site j. If the parking is already saturated at time t_m, then things are obvious. Else, we have $\xi_{t_{m+1}}(i) = \xi_{t_m}(i)$ for $i \neq j\,[\text{mod } n]$ and $\xi_{t_{m+1}}(i) = \xi_{t_m}(i) + 1$ otherwise. We realize after a moment of though that this change in the path corresponds precisely to parking the car that has just arrived at j on the closest vacant site at its right, which is our first claim.

It follows that whenever the parking is not saturated at time t, the first vacant sites at the left and at the right of 0 can be expressed respectively as

$$G_t = \max\{j \leq 0 : \underline{S}_t(j-1) > S_t(j)\}$$

and

$$D_t = \min\{j \geq 0 : S_t(j) < \underline{S}_t(j-1)\}\,.$$

It is easily checked that these quantities can be re-expressed as in the statement. $\square$

2.2 Comparison with simpler parkings

An important source of difficulties in the study of fire models is the lack of monotonicity of the dynamics. In the present case, it is not true in general that adding a few cars or suppressing some Molotov cocktails would increase the occupation of the parking. For instance, adding one car may induce the merging of two neighboring occupied intervals into a larger one which can then be entirely destroyed by a single Molotov cocktail, while the latter would have shattered only one of the two genuine intervals if no car had parked in between to connect them.

A few elementary and intuitively obvious comparisons are nonetheless possible and will be useful in the present study. More precisely, we shall first compare the occupation process Θ with that of the parking driven by the same point process of car arrivals $C = (C_t, t \geq 0)$ but without fires, that is for $M_t \equiv \varnothing$. The latter occupation process will be denoted by $\Theta' = (\Theta'_t, t \geq 0)$, and more generally a prime in the notation such as $X' = X(C, \varnothing)$ will be used for the analogue of $X = X(C, M)$ when Molotov cocktails are suppressed. For instance, for every $j \in \mathbb{Z}$ and $t \geq 0$, $\xi'_t(j)$ denotes the number of cars that have arrived at the site $j\,[\mathrm{mod}\,n]$ before time $t \wedge T'$, where $T' = \inf\{s \geq 0 : \Theta'_s = \mathbf{1}\}$ denotes the saturation time (i.e. the arrival time of the n-th car). For $t \leq T'$, the nonnegative quantity

$$\delta_t(j) = \xi'_t(j) - \xi_t(j)$$

represents the number of cars that arrived at $j\,[\mathrm{mod}\,n]$ before time t and have been burnt in the dynamics with fires; this quantity will play an important role in the sequel.

The following observation should be intuitively obvious.

Lemma 2 *For every $t \geq 0$, each occupied interval of Θ_t is contained into some occupied interval of Θ'_t.*

Proof: With no loss of generality, we may assume that the parking Θ'_t is not yet saturated at time t as otherwise the statement is plain. If we define

$$\Delta_t(0) = 0 \quad \text{and} \quad \Delta_t(j) - \Delta_t(j-1) = \delta_t(j) \qquad \text{for every } j \in \mathbb{Z}\,,$$

then $\Delta_t : \mathbb{Z} \to \mathbb{Z}$ is a non-decreasing quasi-periodic path and, in the obvious notation, $S_t = S'_t - \Delta_t$. This implies that, if the path S'_t reaches a new minimum at j, then so does S_t. By Lemma 1, this shows that $\Theta_t \leq \Theta'_t$, which in turn entails our claim. □

We shall also need a lower bound for occupied intervals. Consider a time $t < T'$ at which the parking without fires Θ'_t is not yet saturated. Recall that S'_t denotes the quasi-periodic path defined by (1) for $M \equiv \varnothing$ and $\underline{S}'_t$ its running minimum.

Lemma 3 *In the notation above, consider first the dynamics without fires. Let I' be an occupied interval at time t and $[j_0, j_1] \subseteq I'$ some arc included in I'. Next consider the dynamics with fires and suppose that the total number of cars that have arrived on I' before time t and have been burnt, is strictly less than the minimum of $S'_t - \underline{S}'_t$ on $[j_0, j_1]$, i.e.*

$$\sum_{j \in I'} \delta_t(j) \; < \; \min_{j_0 \leq j \leq j_1} (S'_t(j) - \underline{S}'_t(j))\,.$$

Then the entire arc $[j_0, j_1]$ is occupied at time t for the dynamics with fires, i.e. $\Theta_t(j) = 1$ for all $j \in [j_0, j_1]$.

Proof: Let g' and d' be the first vacant sites at the left and right of I' in the dynamics without fires, i.e. $I' =]g', d'[$. We use the same notation as in the proof of Lemma 2. In particular, the assumption in the statement reads

$$\Delta_t(d'-1) - \Delta_t(g') = \sum_{j \in I'} \delta_t(j) < \min_{j_0 \le j \le j_1} (S'_t(j) - \underline{S}'_t(j)) .$$

By Lemma 1, we have

$$S'_t(g') = \underline{S}'_t(g') = \underline{S}'_t(d'-1) .$$

We know from Lemma 2 that the site g' is vacant for the dynamics with fires as well, and thus we also have $S_t(g') = \underline{S}_t(g')$. This yields for every $j \in [j_0, j_1]$

$$\begin{aligned} S_t(j) - \underline{S}_t(j-1) &\ge S_t(j) - \underline{S}_t(g') \\ &= S_t(j) - S_t(g') \\ &= S'_t(j) - S'_t(g') - (\Delta_t(j) - \Delta_t(g')) \\ &> S'_t(j) - \underline{S}'_t(g') - \min_{j_0 \le i \le j_1} (S'_t(i) - \underline{S}'_t(i)) . \end{aligned}$$

Because $\underline{S}'_t(i) = \underline{S}'_t(g')$ for every $i \in I'$ and a fortiori for $j_0 \le i \le j_1$, we have

$$\underline{S}'_t(g') + \min_{j_0 \le i \le j_1} (S'_t(i) - \underline{S}'_t(i)) = \min_{j_0 \le i \le j_1} S'_t(i) .$$

We conclude that

$$S_t(j) - \underline{S}_t(j-1) > S'_t(j) - \min_{j_0 \le i \le j_1} S'_t(i) \ge 0$$

and according to Lemma 1, the site j is thus occupied in the dynamics with fires. □

We stress that Molotov cocktails which are thrown outside I' have no impact on the occupation inside I', which is intuitively obvious. Indeed the left and right extremities g' and d' of I' are vacant in the dynamics with fires until time t. They thus serve as barriers which stop the propagation of fires started outside I'.

We will also need lower bounds for the number of cars which are burnt, which will be achieved by comparison with parkings where car arrivals are stopped after a certain time. More precisely, let $0 < s < t$ be two fixed times. Imagine that we stop the arrival of cars after time s, leaving the point process of Molotov cocktails M unchanged. In other words we work with the car arrival process $C'' = (C''_u, u \ge 0)$ defined by $C''_u = C_u$ if $u \le s$ and $C''_u = \varnothing$ if $u > s$. We denote by $B_{s,t}$ the total number of cars which are burnt during the time-interval $(s, t]$ in the original dynamics, i.e.

$$B_{s,t} = \sum_{j \in \mathbb{Z}/n\mathbb{Z}} (\delta_t(j) - \delta_s(j)) ,$$

and by $B''_{s,t}$ the analog quantity for the dynamics in which no cars arrive after time s. The following inequality should be intuitively obvious.

Lemma 4 *In the notation above, we have $B''_{s,t} \leq B_{s,t}$.*

Proof: Consider a car parked at time s at some site $j \in \mathbb{Z}/n\mathbb{Z}$ and let I denote the occupied interval that contains j. If no Molotov cocktails are thrown on I during the time-interval $(s,t]$, then this car will not burn before time t for the dynamics where car arrivals are stopped after time s, i.e. this car does not contribute to $B''_{s,t}$. Otherwise, set $\tilde{t} = \min\{u \in (s,t] : M_u \in I\}$, and follow the evolution of the occupied interval containing I in the original dynamics during the time-interval $(s,\tilde{t}]$. If no car parks at the boundary of I before $\tilde{t}$, then this interval remains unchanged until time $\tilde{t}$ at which is entirely destroyed. Else, let $\tilde{s} < \tilde{t}$ be the first instant after s at which a car parks on the boundary of I. At this instant, the occupied interval containing I increases; denote it by $\tilde{I} \supset I$. Clearly $\min\{u \in (\tilde{s},t] : M_u \in \tilde{I}\} \leq \tilde{t}$ and by induction we see that all the cars that occupied the interval $\tilde{I}$ at time s have been burnt at time $\tilde{t}$. In other words, every car which is burned during the time-interval $(s,t]$ in the dynamics where the car arrivals are stopped after time s is also burned in the original dynamics. $\square$

3 Asymptotics in absence of fires

In this section, we consider the situation when cars arrive at unit rate on each site and independently one of the others and no Molotov are thrown. That is we assume that $(C_t, t \geq 0)$ is a Poisson point process on $\mathbb{Z}/n\mathbb{Z}$ with unit intensity per site and unit of time and $M \equiv \varnothing$. Our goal is to get sharp information about the asymptotic behavior of occupied intervals when the size of the parking $n \to \infty$ and at a time close to saturation. Such questions have been considered previously by several authors. In particular, Pittel [15] observed that in the regime when the number k of vacant sites fulfills $k \sim cn$ for some $0 < c < 1$, then the size of the largest occupied interval is of order $\ln n$ (we stress that Pittel established a much shaper result). Later on, Chassaing and Louchard [11] proved that a phase transition occurs in the regime when $k \sim c\sqrt{n}$; more precisely macroscopic occupied intervals of size of order n appear precisely in this regime. However these results are not sufficient for our purpose as we will need information for intermediate regimes, typically when the number k of vacant sites is such that $\sqrt{n} \ll k \ll n$.

Informally, we may expect the vacant sites to be roughly uniformly spread on the parking; therefore the length of a typical occupied interval should be of order n/k. However this informal analysis may be misleading. Indeed, the distribution of the length of the occupied interval that contains a typical site (say, 0) is related to that of an occupied interval chosen uniformly at random, say L' (remember that primes refer to the dynamics with $M \equiv \varnothing$), by a size-biased transformation. It turns out that when $\sqrt{n} \ll k \ll n$, the variance of L' is much larger than its mean. As a matter of fact, we will see that the lenght of the occupied interval that contains a typical site is rather of order $(n/k)^2$.

To keep track of the size of the parking, we introduce an additional index n in the notation. For every integer m, denote the arrival time of the m-th car by

$$\Gamma_{n,m} = \inf\{t \geq 0 : \#C_{n,t} = m\} \quad \text{with} \quad \#C_{n,t} = \sum_{0 \leq s \leq t} \mathbf{1}_{C_{n,s} \neq \varnothing} \, ;$$

in particular the increments $\Gamma_{n,m+1} - \Gamma_{n,m}$ are i.i.d. exponential variables with mean $1/n$. We also write $(S'_{n,m}(j), j \in \mathbb{Z})$ for the quasi-periodic path defined in (1) for the time $t = \Gamma_{n,m}$.

In our analysis, we shall repeatedly use the following description of the law of $S'_{n,m}$. Let $\xi'_{n,m}(j)$ denote the total number of cars that have arrived at $j \,[\mathrm{mod}\, n]$ until the arrival of the m-th car, so the n-tuple $(\xi'_{n,m}(j), 0 \leq j \leq n-1)$ has the multinomial distribution with parameters m and $(1/n, \ldots, 1/n)$. It is well-known that the latter also arises as the law of an n-tuple of i.i.d. standard Poisson variables and conditioned to have total sum equal to m. Hence, consider a standard Poisson process $(N_t, t \geq 0)$ and write $N^c_t = N_t - t$ for its compensated version. Then $(S'_{n,m}(j), 0 \leq j \leq n)$ has the same distribution as the random walk $(N^c_j, 0 \leq j \leq n)$ conditioned on $N^c_n = -k$.

3.1 Uniform bound for the mean length of an occupied interval

It will often be convenient to use the notation $k = n - m$ for the number of vacant sites just after the arrival of the m-th car.

We denote by $G'_{n,m}$, respectively $D'_{n,m}$, for the first vacant site at the left, respectively the right, of 0 at time $\Gamma_{n,m}$, so $L'_{n,m} = (D'_{n,m} - G'_{n,m} - 1)^+$ is the length of the occupied interval that contains the site 0 when there remain exactly $k = n - m$ vacant sites. The purpose of this section is to establish the following uniform upper bound for the mean length.

Proposition 1 *There is a numerical constant $c > 0$ such that for every $0 \leq m < n$*

$$\mathbb{E}(L'_{n,m}) \leq c \left(\frac{n}{n-m} \right)^2 .$$

The mass distribution of $L'_{n,m}$ is known explicitly; see e.g. Equation (2.6) in Chassaing and Louchard [11]. However the expression is rather involved and as we have not been able to establish Proposition 1 by direct calculations, we shall use a different route that relies on special properties of Borel-Tanner distributions.

To start with, recall that for $0 < s \leq 1$, the Borel distribution with parameter s is the probability measure on positive integers induced by the masses

$$\frac{\mathrm{e}^{-s\ell}(s\ell)^{\ell-1}}{\ell!}, \qquad \ell \geq 1 .$$

We shall denote by β_s a Borel(s) variable; it is known that

$$\mathbb{E}(\beta_s) = \frac{1}{1-s} \quad \text{and} \quad \mathbb{E}(\beta_s^2) = \frac{1}{(1-s)^3} .$$

The sum of k independent Borel(s) variables has the Borel-Tanner law with parameter (s,k), its mass distribution is given by

$$b_{s,k}(\ell) = \frac{k}{\ell(\ell-k)!}(s\ell)^{\ell-k}\mathrm{e}^{-s\ell}, \qquad \ell \geq k .$$

Rather than working directly with the occupied interval that contains some given site, we shall consider the spacing between consecutive vacant sites. Recall that there are k vacant sites on the parking; we pick one of them uniformly at random, denote it by v_1, and then by $v_2, \ldots, v_k$ the remaining ones listed according to the cyclic order. The intervals $[v_i, v_{i+1}[$ for $i = 1, \ldots, k$ (with the convention that $v_{k+1} = v_1$) form a partition of $\mathbb{Z}/n\mathbb{Z}$; we write $\sigma_1, \ldots, \sigma_k$ for the sequence of their sizes.

Lemma 5 *Under the assumptions and notations above, $(\sigma_1, \ldots, \sigma_k)$ has the same distribution as the k-tuple formed by k i.i.d.* Borel(s) *variables conditioned on having total sum equal to n, where the parameter s can be chosen arbitrarily in* $(0,1]$.

Remark. One can also establish a slightly weaker version of Lemma 5 using the correspondence between parking schemes and the additive coalescence (cf. Chassaing and Louchard [11]), and an observation due to Pavlov, see e.g. Corollary 5.8 in [4].

Proof: This result should belong to the folklore of parking schemes, but as we have been unable to spot at a precise reference in the literature, we shall sketch the proof for the reader's convenience. It is a classical (and easy to check) property of Borel variables that the distribution of the k-tuple formed by k i.i.d. Borel(s) variables conditioned on having sum n does not depend on the parameter $s \in (0,1]$, so it suffices to establish the statement in the special case $s = 1$.

Recall that $(S'_j, 0 \leq j \leq n)$ has the same law as the compensated Poissonian walk $(N^c_j, 0 \leq j \leq n)$ conditioned on $N^c_n = -k$. An easy application of the ballot theorem then entails that the cyclic permutation of $(S'_j, 0 \leq j \leq n)$ at the first instant v_1 at which it attains an independent random variable which has the uniform distribution on $\{-k, \ldots, -1\}$, has the same law as a first-passage bridge for the random walk N^c. More precisely $(S'_{v_1+j} - S'_{v_1}, 0 \le j \le n)$ has the same law as $(N^c_j, 0 \le j \le n)$ conditioned on $\inf\{j \ge 0 : N^c_j = k\} = n$. See for instance Theorem 1 in [5] for details. As it is well-known that the durations of the excursions of the compensated Poissonian random walk N^c above its running minimum form a sequence of i.i.d. variables with the Borel(1) law, Lemma 1 yields our claim. □

Note that if L'_i is the length of the occupied interval with left exterior boundary v_i (i.e. v_i is the first vacant site at the left of that interval), then $L'_i = \sigma_i - 1$, and an immediate argument of rotational invariance yields the identity

$$\mathbb{E}(L'_{n,m}) = \frac{1}{n}\mathbb{E}\left(\sum_{i=1}^{k} \sigma_i(\sigma_i - 1)\right) = \frac{k}{n}\mathbb{E}(\sigma_1^2 - \sigma_1)\,. \tag{2}$$

We are now able to tackle the proof of Proposition 1, relying on (2), Lemma 5 and properties of Borel-Tanner distributions.

Proof of Proposition 1: We may suppose without loss of generality that $k^2 \ge 10n$ since otherwise the statement is obvious (because $L'_{n,m} \le n$). Further, it suffices to establish the bound for $k \le n/2$ as clearly the size $L'_{n,k}$ of the occupied interval containing 0 increases when the number of vacant sites k decreases, and hence

$$\mathbb{E}(L'_{n,k}) \le \mathbb{E}(L'_{n,\lfloor n/2 \rfloor}) \le 9c \qquad \text{for every } k \ge n/2\,,$$

provided that the bound stated in Proposition 1 holds for $k = \lfloor n/2 \rfloor$.

By Lemma 5 and (2) we have

$$\mathbb{E}(L'_{n,m}) \le \frac{k}{n}\mathbb{E}(\sigma_1^2) = \frac{k}{n}\mathbb{E}\left(\beta_s^2\,\frac{b_{s,k-1}(n - \beta_s)}{b_{s,k}(n)}\right)\,,$$

where β_s stands for a Borel(s) variable and we agree that $b_{s,k-1}(\ell) = 0$ for $\ell < k - 1$. Set $b^*_{s,k} = \max_\ell b_{s,k}(\ell)$, and observe from the additivity property of the Borel-Tanner distributions that

$$\mathrm{e}^{-1} b_{s,k-1}(\ell - 1) \le b_{s,1}(1) b_{s,k-1}(\ell - 1) \le b_{s,k}(\ell)\,,$$

which yields inequality

$$\frac{b^*_{s,k-1}}{b^*_{s,k}} \le \mathrm{e}\,.$$

We now chose the parameter s such that $b^*_{s,k} = b_{s,k}(n)$, and we get

$$\mathbb{E}(L'_{n,m}) \le \frac{k}{n}\mathrm{e}\,\mathbb{E}(\beta_s^2) = \frac{k\mathrm{e}}{n(1-s)^3}\,. \tag{3}$$

So we need to estimate the value of the parameter s such that the mass function of the Borel-Tanner distribution $\ell \to b_{s,k}(\ell)$ is maximal at $\ell = n$. In this direction, it is convenient to think of the integer variable ℓ as a real one. Taking a logarithmic derivative, we are led to solving the equation

$$-\frac{1}{n} - s + \ln(sn) + \frac{n-k}{n} - \psi(n-k+1) = 0$$

where ψ denotes the Digamma function, that is the logarithmic derivative of the Gamma function. It is well-known that

$$\psi(x+1) = \ln x + \frac{1}{2x} + O(x^{-2}) \qquad \text{as } x \to \infty\,, \tag{4}$$

so the preceding equation can be rewritten in the form

$$s - \ln s = 1 - \frac{k+1}{n} - \ln(1-k/n) - \left(\frac{1}{2(n-k)} + O((n-k)^{-2})\right).$$

Next, elementary calculations yields

$$\begin{aligned}
1 - \frac{k}{3n} - \ln\left(1 - \frac{k}{3n}\right) &= 1 - \frac{k}{n} + \frac{2k}{3n} - \ln\left(1 - \frac{k}{n} + \frac{2k}{3n}\right) \\
&\leq 1 - \frac{k}{n} + \frac{2k}{3n} - \ln\left(1 - \frac{k}{n}\right) - \frac{2k}{3n} \times \frac{1}{1-k/3n} \\
&= 1 - \frac{k}{n} - \ln\left(1 - \frac{k}{n}\right) - \frac{2k^2}{3n(3n-k)} \\
&\leq 1 - \frac{k}{n} - \ln\left(1 - \frac{k}{n}\right) - \frac{20}{9n}\,,
\end{aligned}$$

where in the last line we used the assumption that $k^2 \geq 10n$.

Recall that $k \leq n/2$ and let n be sufficiently large so that

$$\frac{1}{n} + \frac{1}{2(n-k)} + O((n-k)^{-2}) \leq \frac{20}{9n}\,,$$

where $O(\cdot)$ is the function which appears in (4). The calculation above shows that

$$s - \ln s \geq 1 - \frac{k}{3n} - \ln\left(1 - \frac{k}{3n}\right),$$

and as the function $t \to t - \ln t$ is non-increasing on $(0,1)$, we conclude that $s \leq 1 - k/3n$.

Returning to (3), we have thus shown

$$\mathbb{E}(L'_{n,m}) \le 27\mathrm{e}\frac{n^2}{k^2}$$

whenever n is sufficiently large. $\square$

We conclude this section with an easy consequence of Lemma 5 which will be useful to establish a property of propagation of chaos. Recall that $\sigma_1, \dots, \sigma_k$ denote the sequence of spacings between consecutive vacant sites when there remain exactly k vacant sites. We are interested in the sub-sequence obtained by removing the spacing that contains a site chosen uniformly at random, that is one of the spacings picked by size-biased sampling. This means that we consider a random index $j^* \in \{1, \dots, k\}$ with conditional distribution

$$\mathbb{P}(j^* = i \mid \sigma_1, \dots, \sigma_k) = n^{-1}\sigma_i \qquad \text{for } i = 1, \dots, k\,,$$

and the sub-sequence

$$(\tilde{\sigma}_1, \dots, \tilde{\sigma}_{k-1}) = (\sigma_1, \dots, \sigma_{j^*-1}, \sigma_{j^*+1}, \dots, \sigma_k)\,.$$

Corollary 1 *For every $\ell \in \{1, \dots, n-k+1\}$, the conditional distribution of $(\tilde{\sigma}_1, \dots, \tilde{\sigma}_{k-1})$ given $\sigma_{j^*} = \ell$ is that of the sequence of the spacings in the parking $\mathbb{Z}/(n-\ell)\mathbb{Z}$ with Poissonian car arrival and no fires when there remain exactly $k-1$ vacant sites.*

Proof: Let $i_1, \dots, i_{k-1}$ be a sequence of positive integers with $i_1 + \dots + i_{k-1} = n - \ell$. By Lemma 5, we have for every $j = 1, \dots, k$ and $\ell \in \{1, \dots, n-k+1\}$

$$\begin{aligned} & \mathbb{P}\left((\tilde{\sigma}_1, \dots, \tilde{\sigma}_{k-1}) = (i_1, \dots, i_{k-1}), j^* = j, \sigma_{j^*} = \ell\right) \\ = \ & \mathbb{P}\left((\sigma_1, \dots, \sigma_k) = (i_1, \dots, i_{j-1}, \ell, i_j, \dots, i_{k-1}), j^* = j\right) \\ = \ & \frac{\ell}{n} \times \frac{b_{s,1}(\ell)}{b_{s,k}(n)} \prod_{r=1}^{k-1} b_{s,1}(i_r)\,. \end{aligned}$$

Summing over the possible values for j, we deduce that

$$\begin{aligned} & \mathbb{P}\left((\tilde{\sigma}_1, \dots, \tilde{\sigma}_{k-1}) = (i_1, \dots, i_{k-1}), \sigma_{j^*} = \ell\right) \\ = \ & \frac{k\ell}{n} \, \frac{b_{s,1}(\ell)}{b_{s,k}(n)} \prod_{r=1}^{k-1} b_{s,1}(i_r) \\ = \ & \frac{k\ell}{n} \, \frac{b_{s,1}(\ell) b_{s,k-1}(n-\ell)}{b_{s,k}(n)} \times \frac{1}{b_{s,k-1}(n-\ell)} \prod_{r=1}^{k-1} b_{s,1}(i_r)\,. \end{aligned}$$

Note that the second term in the product above corresponds to the mass distribution of the

$(k-1)$-tuple formed by i.i.d. Borel(s) variables conditioned on having sum $n-\ell$. Our claim follows from the comparison with Lemma 5. $\square$

3.2 Brownian limits

Proposition 1 provides a useful uniform upper bound for the expected length of occupied intervals. However such information is not sufficient for our purpose, and we will also need precise estimates of these lengths in certain regimes. We first analyze the asymptotic behavior of the path that encodes the occupation of the parking when the number of sites tends to infinity.

Proposition 2 *Fix arbitrary $a<b$. In the regime when $n,k\to\infty$ with $n^{2/3}\ll k\ll n$, the rescaled path*

$$\frac{k}{n}S'_{n,n-k}(\lfloor n^2k^{-2}u\rfloor)\,,\quad a\le u\le b$$

converges in distribution on the space of càdlàg paths on $[a,b]$ endowed with the maximum norm towards

$$W_u-u\,,\quad a\le u\le b\,,$$

where $(W_u,u\in\mathbb{R})$ is a standard two-sided Brownian motion.

For the sake of simplicity, we shall establish Proposition 2 for $a=0$ and $b=1$, the case of arbitrary a and b only requiring a heavier notation. The proof relies on a technical asymptotic property of Poisson mass-distributions

$$p_n(\ell)=\frac{\mathrm{e}^{-n}n^\ell}{\ell!}\,,\qquad \ell\in\mathbb{Z}_+\,,$$

which we know state.

Lemma 6 *Consider two sequences of nonnegative integers $(k_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ such that $n^{2/3}\ll k_n\ll n$ and*

$$\lim_{n\to\infty}\left(\frac{k_n}{n}x_n-\frac{n}{k_n}\right)=w$$

for some $w\in\mathbb{R}$. Then we have

$$\lim_{n\to\infty}\frac{p_{n-\lfloor n^2/k_n^2\rfloor}(n-k_n-x_n)}{p_n(n-k_n)}=\exp(-w-1/2)\,.$$

Proof: We express the ratio as

$$\begin{aligned}\frac{p_{n-\lfloor n^2/k_n^2\rfloor}(n-k_n-x_n)}{p_n(n-k_n)} &= \mathrm{e}^{\lfloor n^2/k_n^2\rfloor} \times \frac{(n-\lfloor n^2/k_n^2\rfloor)^{n-k_n-x_n}}{n^{n-k_n}} \times \frac{(n-k_n)!}{(n-k_n-x_n)!} \\ &= \mathrm{e}^{\lfloor n^2/k_n^2\rfloor} \times \left(1-n^{-1}\lfloor n^2/k_n^2\rfloor\right)^{n-k_n-x_n} \times \prod_{i=0}^{x_n-1}\left(1-\frac{k_n+i}{n}\right).\end{aligned}$$

Recall that $n^2/k_n^3 \ll 1$ and $x_n \sim n^2/k_n^2$. We estimate the logarithm of the preceding quantity and get

$$\begin{aligned}&\left\lfloor\frac{n^2}{k_n^2}\right\rfloor - (n-k_n-x_n)\left(n^{-1}\left\lfloor\frac{n^2}{k_n^2}\right\rfloor + \frac{n^2}{2k_n^4}\right) - \sum_{i=0}^{x_n-1}\left(\frac{k_n+i}{n}\right) - \frac{1}{2}\sum_{i=0}^{x_n-1}\left(\frac{k_n+i}{n}\right)^2 + o(1) \\ =\ &\left\lfloor\frac{n^2}{k_n^2}\right\rfloor - \left(\left\lfloor\frac{n^2}{k_n^2}\right\rfloor + \frac{n^3}{2k_n^4} - \frac{n}{k_n} - x_n\frac{n}{k_n^2}\right) - \left(x_n\frac{k_n}{n} - \frac{x_n^2-x_n}{2n}\right) - \frac{1}{2}x_n\frac{k_n^2}{n^2} + o(1)\,.\end{aligned}$$

After some simplifications using the identity $x_nk_n/n = n/k_n + w + o(1)$, we see that the above quantity can be expressed as $-w-1/2+o(1)$, which yields our claim. □

We can now proceed with the proof of Proposition 2.

Proof of Proposition 2: Recall that $(N_u^c = N_u - u, u \geq 0)$ is a compensated Poisson process, and that

$$\left(\frac{k}{n}S'_{n,m}(\lfloor n^2k^{-2}u\rfloor)\,, 0 \leq u \leq 1\right)$$

has the same law as the process $\left(\frac{k}{n}N^c_{\lfloor n^2k^{-2}u\rfloor}\,, 0 \leq u \leq 1\right)$ conditioned on $N_n^c = -k$, i.e. on $N_n = m = n-k$.

Consider a continuous functional Φ on the space of càdlàg paths on the unit interval, with values in $[0,1]$. Observe that $\mathbb{P}(N_\ell^c = j) = p_\ell(j+\ell)$, so an application of the Markov property for the compensated Poisson process at time n^2/k^2 yields

$$\begin{aligned}&\mathbb{E}\left(\Phi\left(\frac{k}{n}S'_{n,m}(\lfloor n^2k^{-2}u\rfloor)\,, 0 \leq u \leq 1\right)\right) \\ =\ &\mathbb{E}\left(\Phi\left(\frac{k}{n}N^c_{\lfloor n^2k^{-2}u\rfloor}\,, 0 \leq u \leq 1\right)\frac{p_{n-\lfloor n^2/k^2\rfloor}(n-k-N_{\lfloor n^2/k^2\rfloor})}{p_n(n-k)}\right)\end{aligned}$$

We next let $k = k_n$ depend on n with $k_n \gg n^{2/3}$. By Donsker's invariance principle,

$$\left(\frac{k_n}{n}N^c_{\lfloor n^2k_n^{-2}u\rfloor}\,, 0 \leq u \leq 1\right)$$

converges weakly as $n \to \infty$ to a standard Brownian motion on the unit time interval, $(W_u, 0 \leq u \leq 1)$. It then follows from Lemma 6 and Fatou's Lemma that

$$\begin{aligned}\liminf_{n\to\infty} \mathbb{E}\left(\Phi\left(\frac{k_n}{n}S'_{n,n-k_n}(\lfloor n^2 k_n^{-2} u\rfloor)\,, 0 \leq u \leq 1\right)\right) &\geq \mathbb{E}\left(\Phi\left(W_u\,, 0 \leq u \leq 1\right)\mathrm{e}^{-W_1-1/2}\right) \\ &= \mathbb{E}\left(\Phi\left(W_u - u\,, 0 \leq u \leq 1\right)\right)\end{aligned}$$

where the last identity stems from the classical relation of absolute continuity between the law of the Brownian motion with drift and the Wiener measure. Replacing Φ by $1-\Phi$, we get the converse upper-bound

$$\limsup_{n\to\infty} \mathbb{E}\left(\Phi\left(\frac{k_n}{n}S'_{n,n-k_n}(\lfloor n^2 k_n^{-2} u\rfloor)\,, 0 \leq u \leq 1\right)\right) \leq \mathbb{E}\left(\Phi\left(W_u - u\,, 0 \leq u \leq 1\right)\right)\,,$$

which completes the proof of (i) in the case $a = 0$ and $b = 1$. The general case can be proven by the same argument, but with a heavier notation. $\square$

Proposition 2 enables us to investigate precisely the asymptotics the occupied interval containing 0 in the regime of interest. Indeed, it suggests that after proper rescaling, the occupied interval should converge weakly to the interval straddling 0 during which a two-sided Brownian motion with negative drift makes an excursion above its running minimum. To give a precise statement, it is convenient to introduce some further notation. Recall that $(W_u, u \in \mathbb{R})$ denotes a two-sided Brownian motion; we write

$$R_u = W_u - u - \min_{v \leq u}(W_v - v)$$

for the Brownian motion with negative drift reflected at its running minimum. We set

$$G = \sup\{u < 0 : R_u = 0\} \quad \text{ and } \quad D = \inf\{u > 0 : R_u = 0\}\,,$$

so that (G, D) is the excursion interval of the reflected process away from 0 that straddles the origin. We also denote by

$$X_u = \begin{cases} 0 & \text{if} \quad u \leq G \\ R_u & \text{if} \quad G < u < D \\ 0 & \text{if} \quad u \geq D \end{cases} \tag{5}$$

the corresponding excursion. Similarly, we introduce for every positive integers $m < n$ the periodic reflected path $R'_{n,m}(j) = S'_{n,m}(j) - \underline{S}'_{n,m}(j)$ and finally we set

$$X'_{n,m}(j) = \begin{cases} 0 & \text{if} \quad j \leq G'_{n,m} \\ R'_{n,m}(j) & \text{if} \quad G'_{n,m} < j < D'_{n,m} \\ 0 & \text{if} \quad j \geq D'_{n,m} \end{cases} \tag{6}$$

where $G'_{n,m}$ and $D'_{n,m}$ denote the first vacant site at the left, respectively the right, of 0 at time $\Gamma_{n,m}$, viz.

$$\begin{aligned} G'_{n,m} &= \max\{j \leq 0 : \underline{S}'_{n,m}(j-1) > S'_{n,m}(j)\} \\ D'_{n,m} &= \min\{j \geq 0 : \underline{S}'_{n,m}(j-1) < S'_{n,m}(j)\} . \end{aligned}$$

Proposition 3 *In the regime when $n, k \to \infty$ with $n^{2/3} \ll k \ll n$, the rescaled path*

$$\left(\frac{k}{n} X'_{n,n-k}(\lfloor n^2 k^{-2} u \rfloor) \, , u \in \mathbb{R} \right)$$

converges in distribution on the space of càdlàg paths endowed with the maximum norm towards $(X_u, u \in \mathbb{R})$. In particular the pair of rescaled extremities of the occupied interval straddling 0,

$$\left(\frac{k^2}{n^2} G'_{n,n-k}, \frac{k^2}{n^2} D'_{n,n-k} \right)$$

converges in distribution towards a pair of random variables with joint distribution

$$\frac{1}{\sqrt{2\pi(y-x)^3}} \exp(-(y-x)/2) \mathrm{d}x \mathrm{d}y \, , \qquad x < 0 < y \, .$$

The difficulty in deriving rigorously this weak limit from Proposition 2 is that the convergence stated there only concerns paths on a compact interval, whereas in order to investigate the left-extremity G' of the occupied interval, we have to deal with the location of the overall minimum of the path on $(-\infty, 0]$ (the right-extremity D' is much easier to handle once the location G' and more precisely the value of the minimum $W_{G'} - G'$ are known). To resolve this difficulty, we shall need an *a priori* stochastic bound for the left-extremity G', which in turn relies on the following technical lemma. Recall that $p_n(\cdot)$ denotes the mass-distribution of the Poisson law with parameter n.

Lemma 7 *For every $b > 0$, in the regime $k, n \to \infty$ with $n^{2/3} \ll k \ll n$, we have*

$$\max_{bn^2/k^2 \leq \ell \leq n-k} \frac{p_n(n-\ell-k)}{p_n(n-k)} \longrightarrow \mathrm{e}^{-b/2} \, .$$

Proof: Without loss of generality, we may suppose that b is rational. We start by observing that

$$\lim \frac{p_n(n-k-bn^2/k^2)}{p_n(n-k)} = \mathrm{e}^{-b/2}\,, \tag{7}$$

where the limit is taken as $n, k \to \infty$ in the regime $k \gg n^{2/3}$ and $bn^2/k^2 \in \mathbb{N}$. Indeed

$$\frac{p_n(n-k-bn^2/k^2)}{p_n(n-k)} = \mathrm{e}^{bn^2/k^2}\left(1-bn/k^2\right)^{n-k-bn^2/k^2}\prod_{i=0}^{bn^2/k^2-1}\left(1-\frac{k+i}{n}\right),$$

and the logarithm of this quantity can be expressed as

$$\begin{aligned}
&\frac{bn^2}{k^2}-\left(n-k-\frac{bn^2}{k^2}\right)\left(\frac{bn}{k^2}+\frac{b^2n^2}{2k^4}\right)-\sum_{i=0}^{bn^2k^{-2}-1}\left(\frac{k+i}{n}+\frac{1}{2}\left(\frac{k+i}{n}\right)^2\right)+o(1)\\
=\ &\frac{bn^2}{k^2}-\left(\frac{bn^2}{k^2}-\frac{bn}{k}-\frac{b^2n^3}{k^4}+\frac{b^2n^3}{2k^4}\right)-\left(\frac{bn}{k}+\frac{b^2n^3}{2k^4}+\frac{b}{2}\right)+o(1)\\
=\ &-\frac{b}{2}+o(1)\,.
\end{aligned}$$

Next, consider the function

$$x \mapsto p_x(x-k) = \mathrm{e}^{-x}x^{x-k}/\Gamma(x-k+1)\,,$$

and view now the variable x as a positive real number. Take the logarithmic derivative; we get

$$x \mapsto \ln x - k/x - \psi(x-k+1)\,,$$

where ψ denotes the Digamma function. Using the estimate (4), we can re-express this quantity as

$$\begin{aligned}
&-\ln\left(1-\frac{k}{x}\right)-\frac{k}{x}-\frac{1}{2(x-k)}+O((x-k)^{-2})\\
\geq\ &\frac{k^2}{2x^2}-\frac{1}{2(x-k)}+O((x-k)^{-2})
\end{aligned}$$

and it is easily checked that this is positive on $[3k/2, k^2/2]$ for k is sufficiently large. Therefore, since $k^2 \gg n$, we have

$$\max_{bn^2/k^2\leq \ell\leq n-3k/2} p_n(n-\ell-k) = p_n(\lfloor n-bn^2/k^2\rfloor - k)\,. \tag{8}$$

Finally, an elementary calculation based on Stirling formula yields the estimate

$$\begin{aligned}\ln p_n(n-k) &= -k-(n-k)\ln(1-k/n)-\frac{1}{2}\ln n+O(1)\\ &= -\sum_{\ell=2}^{\infty}\frac{1}{\ell(\ell-1)}k^{\ell}n^{1-\ell}-\frac{1}{2}\ln n+O(1)\,,\end{aligned}$$

and since $k \ll n$, we deduce that

$$\max_{n-3k/2\le \ell\le n-k} p_n(n-\ell-k)=o(p_n(n-k))\,.$$

Combining this with (7) and (8) completes the proof. □

We now proceed with the proof of Proposition 3.

Proof of Proposition 3: It is convenient to introduce the dual compensated Poisson process $\check{N}^c_s = s - N_s$, where $(N_s, s \geq 0)$ is a standard Poisson process. The reversed path $(S'_{n,n-k}(-\ell), 0 \leq \ell \leq n)$ has then the same distribution as $(\check{N}^c_\ell, 0 \leq \ell \leq n)$ conditionally on $\check{N}^c_n = k$. Fix $b > 0$ and observe that if $-G'_{n,n-k} > bn^2/k^2$, then necessarily the reversed path $S'_{n,n-k}(-\ell)$ visits 0 again for some $\ell > bn^2/k^2$. This yields the bound

$$\mathbb{P}(-G'_{n,n-k}\geq bn^2/k^2)\leq \mathbb{P}(\check{N}^c_\ell=0 \text{ for some } bn^2/k^2\leq \ell<n \mid \check{N}^c_n=k)\,.$$

Applying the strong Markov property of the random walk $\check{N}^c$ at its first return to 0 after bn^2/k^2, we get

$$\begin{aligned}\mathbb{P}(-G'_{n,n-k}\geq bn^2/k^2) &\leq \max_{bn^2/k^2\leq\ell\leq n-k}\mathbb{P}(\check{N}^c_{n-\ell}=k)/\mathbb{P}(\check{N}^c_n=k)\\ &= \max_{bn^2/k^2\leq\ell\leq n-k}\frac{p_n(n-\ell-k)}{p_n(n-k)}\,.\end{aligned}$$

It follows from Lemma 7 that in the regime $k, n \to \infty$ with $n^{2/3} \ll k \ll n$

$$\limsup \mathbb{P}(-G'_{n,n-k}\geq bn^2/k^2)\leq \mathrm{e}^{-b/2}\,.$$

This stochastic bound implies that when b is large and $n, k \to \infty$ in the preceding regime, then with high probability the location and value of the overall minimum of $S'_{n,n-k}$ on $(-\infty, 0]$ are the same as on $[-bn^2/k^2, 0]$. On the other hand, the location and value of the overall minimum of $W_u - u$ on $(-\infty, 0]$ are also the same as on $[-b, 0]$ with high probability when b is large. We can then deduce the first claim of Proposition 3 from Proposition 2 by routine arguments. Finally, that (G, D) has the distribution given in the statement belongs to the Brownian folklore. □

We can now deduce from Proposition 3 the asymptotic behavior of the empirical distribution of the length of occupied intervals by standard arguments involving rotational invariance and propagation of chaos, see [18].

Corollary 2 *For every $j \in \mathbb{Z}/n\mathbb{Z}$, denote by $L'_{n,m}(j)$ the length of the occupied interval which contains the site j just after the arrival of the m-th car (by convention $L'_{n,m}(j) = 0$ is the site j is vacant at that time). Consider the empirical distribution*

$$\mu'_{n,m} = \frac{1}{n} \sum_{j \in \mathbb{Z}/n\mathbb{Z}} \delta_{(n-m)^2 n^{-2} L'_{n,m}(j)} .$$

Then $\mu'_{n,m}$ converges in probability on the space of probability measures on $[0, \infty)$ endowed with Prohorov's distance as $n, m \to \infty$ in the regime $n^{2/3} \ll n - m \ll n$, towards

$$\mu(\mathrm{d}x) = \frac{1}{\sqrt{2\pi x}} \exp(-x/2)\mathrm{d}x , \qquad x > 0 .$$

We stress that the result would fail in the regime $n - m \sim \sqrt{n}$, and refer to Theorem 2.1 of Chassaing and Louchard [11] for a different limiting law in the later case.

Proof: Consider a continuous bounded function $f : \mathbb{R}_+ \to \mathbb{R}$ and set

$$\langle \mu'_{n,m}, f \rangle = \frac{1}{n} \sum_{j \in \mathbb{Z}/n\mathbb{Z}} f((n-m)^2 n^{-2} L'_{n,m}(j))$$

and $\langle \mu, f \rangle = \int f(x)\mu(\mathrm{d}x)$. For every n, we pick two sites in $\mathbb{Z}/n\mathbb{Z}$ uniformly at random, say j^* and $j^\dagger$, and write for simplicity $L'_{n,m}(j^*) = L^*_{n,m}$ and $L'_{n,m}(j^\dagger) = L^\dagger_{n,m}$.

By rotational invariance, $L^*_{n,m}$ and $L^\dagger_{n,m}$ have both the same law as $L'_{n,m} = (D'_{n,m} - G'_{n,m} - 1)^+$, the length of the occupied interval that contains the site 0. As a consequence, we get from Proposition 3(ii) that

$$\begin{aligned}
\mathbb{E}(\langle \mu'_{n,m}, f \rangle) &= \mathbb{E}(f((n-m)^2 n^{-2} L^*_{n,m})) \\
&= \mathbb{E}(f((n-m)^2 n^{-2} (D'_{n,m} - G'_{n,m} - 1)^+)) \\
&\to \int_{-\infty}^{0} \mathrm{d}x \int_{0}^{\infty} \mathrm{d}y f(y-x) \frac{1}{\sqrt{2\pi (y-x)^3}} \exp(-(y-x)/2) \\
&= \langle \mu, f \rangle .
\end{aligned}$$

Next we can express the second moment of $\langle \mu'_{n,m}, f \rangle$ as

$$\mathbb{E}(\langle \mu'_{n,m}, f \rangle^2) = \mathbb{E}(f((n-m)^2 n^{-2} L^*_{n,m}) f((n-m)^2 n^{-2} L^\dagger_{n,m})) .$$

Because $|j^* - j^\dagger| \gg n^2/(n-m)^2$ with high probability when $n, m \to \infty$ in the regime of concern, the occupied intervals $I^*_{n,m}$ and $I^\dagger_{n,m}$ containing respectively j^* and $j^\dagger$ are disjoint with high probability. Further, it follows from Corollary 1 that conditionally on $j^\dagger \notin I^*_{n,m}$ and $L^*_{n,m} = \ell$, $L^\dagger_{n,m}$ has the same law as $L'_{n-\ell,m-\ell-1}$. It is then routine to deduce that the rescaled lengths $(n-m)^2 n^{-2} L^*_{n,m}$ and $(n-m)^2 n^{-2} L^\dagger_{n,m}$ are asymptotically independent. Hence

$$\mathbb{E}(\langle \mu'_{n,m}, f\rangle^2) \to \langle \mu, f\rangle^2$$

and we conclude that

$$\langle \mu'_{n,m}, f\rangle \to \langle \mu, f\rangle \qquad \text{in } L^2(\mathbb{P}),$$

which yields our claim. □

4 Main results

Throughout the remainder of this paper, we assume that cars arrive at unit rate on each site and independently one of the others, while Molotov cocktails are thrown on each site at rate $n^{-\alpha}$, independently of the arrivals of cars, where α is some parameter in $(0,1)$. In other words, $(C_t, t \geq 0)$ and $(M_t, t \geq 0)$ are two independent Poisson point processes on $\mathbb{Z}/n\mathbb{Z}$ with respective intensities 1 and $n^{-\alpha}$ per site and unit of time.

We are interested in the following quantities. First, for every $n \in \mathbb{N}$, we introduce the first instant when the parking of size n is saturated,

$$T_n := \inf\{t \geq 0 : \Theta_{n,t} = \mathbf{1}\}.$$

Next, for every $t \geq 0$, we denote the mean occupation of the parking with size n at time t by

$$\theta_{n,t} = n^{-1} \sum_{j \in \mathbb{Z}/n\mathbb{Z}} \Theta_{n,t}(j).$$

Our main result claims that for any α, the mean occupation at time t is close to t as long as $t < 1$, with high probability. When $\alpha > 2/3$, the parking becomes saturated at time close to 1. For $\alpha < 2/3$, the mean occupation drops suddenly to nearly 0 right after time 1 although the parking is never fully saturated. Here are the formal statements.

Theorem 1 (i) *For every $0 < t < 1$, we have*

$$\lim_{n\to\infty} \theta_{n,t} = t \qquad \textit{in probability}.$$

(ii) *For* $\alpha > 2/3$, *we have*

$$\lim_{n\to\infty} T_n = 1 \qquad \textit{in probability}.$$

(iii) *For* $\alpha < 2/3$, $1 < t < 2$ *and* $\varepsilon > 0$, *we have*

$$\lim_{n\to\infty} \mathbb{P}(\theta_{n,t} \leq t - 1 + \varepsilon) = 1\,.$$

(iv) *For* $\alpha < 2/3$, *we have for every* $t < 2$

$$\lim_{n\to\infty} \mathbb{P}(T_n \leq t) = 0\,.$$

Remark. The first instant when a Molotov cocktail is thrown after the saturation time is a renewal time for the occupation of the parking. For $\alpha > 2/3$, we know from (ii) that the latter is close to 1 when n is large, and thus (i) can be reinforced as

$$\lim_{n\to\infty} \theta_{n,t} = \{t\} = t - \lfloor t \rfloor \qquad \text{in probability for all } t \geq 0.$$

We conjecture that this asymptotic for the mean occupation holds also when $\alpha \leq 2/3$, but have been unable so far to establish this due to the lack of renewal in that situation. In the same vein, we also conjecture that (iv) holds for all $t \geq 0$.

The rest of this section is devoted to the proof of Theorem 1. The parts (i) and (ii) follows rather easily from the material developed in the preceding sections, while (iii) and (iv) are more delicate.

Recall that $\Gamma_{n,m}$ denotes the arrival time of the m-th car, and that the increments $\Gamma_{n,m+1} - \Gamma_{n,m}$ are i.i.d. exponential variables with mean $1/n$. In particular the mean number of Molotov cocktails that are thrown during the time-interval $[\Gamma_{n,m}, \Gamma_{n,m+1})$ equals $n^{-\alpha}$. For every $j \in \mathbb{Z}/n\mathbb{Z}$, we also denote by $\delta_{n,m}(j)$ the number of cars that have arrived at the site j and have been burnt before the arrival time $\Gamma_{n,m}$ of the m-th car. We first point at the following upper-bound.

Lemma 8 *There is a numerical constant* c *such that for every* $j \in \mathbb{Z}$ *and* $0 \leq m < n$

$$\mathbb{E}(\delta_{n,m}(j)) \leq c\frac{n^{1-\alpha}}{n-m}\,.$$

Proof: Recall that $L'_{n,m}$ denotes the length of the occupied interval that contains the site 0 at time $\Gamma_{n,m}$ when no Molotov cocktails are thrown. By rotational invariance, the distribution of this quantity remains the same if we replace 0 by any other site $j \in \mathbb{Z}/n\mathbb{Z}$. Lemma 2 now shows that the size of each occupied interval which is hit by a Molotov cocktail thrown during the time-interval $[\Gamma_{n,m}, \Gamma_{n,m+1})$ can be stochastically bounded from above by $L'_{n,m}$. Further, we deduce from Proposition 1 that the mean number of cars which are burnt before time $\Gamma_{n,m}$ can bounded from above by

$$\mathbb{E}\left(\sum_{j\in\mathbb{Z}/n\mathbb{Z}} \delta_{n,m}(j)\right) \le n^{-\alpha} c \sum_{\ell=0}^{m} \left(\frac{n}{n-\ell}\right)^2 \le c\frac{n^{2-\alpha}}{n-m}\,.$$

As $\mathbb{E}(\delta_{n,m}(j))$ does not depend of $j \in \mathbb{Z}/n\mathbb{Z}$, this establishes our claim. $\square$

We are now able to establish parts (i) and (ii) of Theorem 1.

Proof of Theorem 1(i): Fix an arbitrarily $t \in (0,1)$ and $0 < 2\varepsilon < 1-t$. Next set $m_n = \lfloor (t+\varepsilon)n \rfloor$ and $m'_n = \lfloor (t-\varepsilon)n \rfloor$. By the law of large numbers, the bounds

$$\Gamma_{n,m'_n} \le t \le \Gamma_{n,m_n}$$

hold with a high probability when n is large. Thanks to Lemma 8, the mean number of cars which have been burnt up to time Γ_{n,m_n} is bounded from above by $c(1-t-\varepsilon)^{-1}n^{1-\alpha}$. By Markov inequality, the number of cars which have been burnt up to time Γ_{n,m_n} can thus be bounded from above by $n^{1-\alpha/2}$ with high probability. Plainly, on the latter event, the mean occupation at time t fulfills

$$t - \varepsilon - n^{-\alpha/2} \le \theta_{n,t} \le t + \varepsilon\,,$$

which proves our claim. $\square$

Proof of Theorem 1(ii): This follows from a variation of the preceding argument. Take $m = n - \lfloor n^\beta \rfloor$ for some $2/3 < \beta < \alpha$, so by Lemma 8 and Markov's inequality, the probability that more than n^β cars have burned at time $\Gamma_{n,m}$ is less than $cn^{2-\alpha-2\beta}$. Thus with high probability, there are at most $2n^\beta$ vacant sites at time time $\Gamma_{n,m}$. Because the mean number of Molotov cocktails that are thrown between times $\Gamma_{n,n-n^\beta}$ and $\Gamma_{n,n+n^\beta}$ is $2n^{-\alpha+\beta} \ll 1$, we conclude that with high probability, the parking is saturated when the $(n+n^\beta)$-th car arrives, and this entails our claim since $\Gamma_{n,n+n^\beta} \sim 1$. $\square$

Our approach to establish Theorem 1 (iii) when $\alpha < 2/3$ consists in showing first that there exist times close to 1 at which the length of the occupied interval containing a typical site is of order n^β for some $\beta > \alpha$. The probability that such an interval is not hit by a Molotov cocktail during a time interval of fixed duration $\varepsilon > 0$ is of order $\exp(-\varepsilon n^{\beta-\alpha}) \ll 1$. This means that the lifetime of these intervals is small, and hence a typical site will be vacant shortly after that time.

In this direction, denote for every $j \in \mathbb{Z}/n\mathbb{Z}$ by $L_{n,m}(j)$ the length of the occupied interval which contains the site j just after the arrival of the m-th car (by convention $L_{n,m}(j) = 0$ is the site j is vacant at that time). We shall investigate the asymptotic behavior of the empirical distribution

$$\mu_{n,m} = \frac{1}{n} \sum_{j \in \mathbb{Z}/n\mathbb{Z}} \delta_{(n-m)^2 n^{-2} L_{n,m}(j)} \,.$$

Roughly speaking we observe that there exists regimes at which the impact of fires is still low (in the sense that the behavior is the same as if there were no fires; cf. Corollary 2) even though the size of typical intervals becomes large, namely greater than n^α.

Proposition 4 *Suppose $\alpha < 2/3$. In the regime $n, m \to \infty$ with*

$$n^{\frac{2}{3} \vee (1-2\alpha/3)} \ln^{4/3} n \ll n - m \ll n \,,$$

$\mu_{n,m}$ converges in probability on the space of probability measures on $[0, \infty)$ endowed with Prohorov's distance towards

$$\mu(\mathrm{d}x) = \frac{1}{\sqrt{2\pi x}} \exp(-x/2) \mathrm{d}x \,, \qquad x > 0 \,.$$

We take Proposition 4 for granted, postponing its proof to the next section, and now establish Theorem 1(iii).

Proof of Theorem 1(iii): Since $\alpha < 2/3$, we may pick $\beta \in (\frac{2}{3} \vee (1-2\alpha/3), 1-\alpha/2)$. Then we set $k_n = n - m_n = \lfloor n^\beta \rfloor$, and we are in the regime of Proposition 4. We work conditionally on the occupation of the parking after the arrival of the m_n-th car. Fix $j \in \mathbb{Z}/n\mathbb{Z}$ and consider the occupied interval that contains j, say I_j. The probability that a Molotov cocktail will be thrown on I_j during a time interval of duration t is $1 - \exp(-L_{n,m_n}(j) t n^{-\alpha})$, where $L_{n,m_n}(j) = |I_j|$ denotes the number of sites in I_j. So, if we consider the dynamics in which car arrivals are stopped after the arrival of the m_n-th car, Γ_{n,m_n}, the conditional mean number of cars which are burnt between times Γ_{n,m_n} and $t + \Gamma_{n,m_n}$ is given by

$$\sum_{j \in \mathbb{Z}/n\mathbb{Z}} (1 - \exp(-L_{n,m_n}(j) t n^{-\alpha})) = n \langle \mu_{n,m_n}, 1 - \exp(-t n^{-\alpha} n^2 k_n^{-2} \cdot) \rangle \,.$$

We pick $0 < \eta < 2 - \alpha - 2\beta$ and take $t = t_n = n^{-\eta}$, so

$$\lim_{n \to \infty} (1 - \exp(-t_n n^{-\alpha} n^2 k_n^{-2} x)) = 1 \qquad \text{for every } x > 0.$$

We deduce from Proposition 4 and the porte-manteau theorem that the conditional mean number of cars that have burnt between Γ_{n,m_n} and $t_n + \Gamma_{n,m_n}$ is asymptotically close to n. We may now invoke Lemma 4 and conclude that for the original dynamics, the mean number of cars which are burnt between Γ_{n,m_n} and $t_n + \Gamma_{n,m_n}$ is asymptotically bounded from below by n.

On the other hand, the total number of cars which have arrived between Γ_{n,m_n} and $t_n+\Gamma_{n,m_n}$ is close to $nt_n = n^{1-\eta} \ll n$. Thus the density of occupation of the parking at time $t_n+\Gamma_{n,m_n}$ must be small with high probability. As $t_n+\Gamma_{n,m_n} \sim 1$ and the increase of the mean occupation on any time-interval with duration s is obviously bounded from above by s, this entails our claim. $\Box$

We next prepare the ground for the proof of Theorem 1(iv). Roughly speaking, we have to check that for $\alpha < 2/3$, destruction of cars by fires at times close to 1 occurs more rapidly than new cars arrive, which prevents the saturation of the parking. In this direction, we consider the following setting. Let $a > 0$ and $(x_i, i \in \mathbb{N})$ be a collection of nonnegative real numbers which may be viewed as masses. We mark each x_i at rate ax_i, independently one of the others, i.e. each x_i receives a mark at time $(ax_i)^{-1}\mathbf{e}_i$ where $(\mathbf{e}_i, i \in \mathbb{N})$ is a sequence of i.i.d. standard variables. For every $s \geq 0$, let

$$X_s = \sum_{i=1}^{\infty} x_i \mathbf{1}_{\mathbf{e}_i \leq ax_i s}$$

be the sum of the masses that have a mark at time s.

Lemma 9 *Let $0 < t_0 < t_1$ and $b > 0$. There is the inequality*

$$\mathbb{P}(X_s \leq bs \text{ for some } t_0 \leq s \leq t_1) \leq (1+\ln(t_1/t_0)) \exp\left(1 - \frac{a}{b\mathrm{e}^3}\sum_{i\in\mathcal{I}} x_i^2\right)$$

where

$$\mathcal{I} = \{i : x_i \leq (bt_0) \wedge (1/at_1)\} .$$

Proof: Observe first that the Laplace transform of X_s is given for every $q > 0$ by

$$\begin{aligned}
\mathbb{E}(\exp(-qX_s)) &= \prod_{i=1}^{\infty} \mathbb{E}(\mathbf{1}_{\mathbf{e}_i > ax_i s} + \mathrm{e}^{-qx_i}\mathbf{1}_{\mathbf{e}_i \leq ax_i s}) \\
&= \prod_{i=1}^{\infty} \left(\mathrm{e}^{-ax_i s} + (1-\mathrm{e}^{-ax_i s})\mathrm{e}^{-qx_i}\right) \\
&= \exp\left(\sum_{i=1}^{\infty} \ln\left(1 - (1-\mathrm{e}^{-ax_i s})(1-\mathrm{e}^{qx_i})\right)\right) \\
&\leq \exp\left(-\sum_{i=1}^{\infty} (1-\mathrm{e}^{-ax_i s})(1-\mathrm{e}^{qx_i})\right) .
\end{aligned}$$

Next, observe from the Markov inequality that

$$\mathbb{P}(X_s \leq \mathrm{e}bs) \leq \mathrm{e}\mathbb{E}\left(\exp\left(-\frac{1}{\mathrm{e}bs}X_s\right)\right) ,$$

and hence

$$\mathbb{P}(X_s \le \mathrm{e}bs) \le \exp\left(1 - \sum_{i=1}^{\infty}(1-\mathrm{e}^{-ax_is})(1-\mathrm{e}^{-x_i/\mathrm{e}bs})\right).$$

Then recall the definition of the set of indices $\mathcal{I}$ in the statement and note that for every $i \in \mathcal{I}$ and $t_0 \le s \le t_1$, we have the bounds

$$1-\mathrm{e}^{-ax_is} \ge \mathrm{e}^{-1}ax_is \quad \text{and} \quad 1-\mathrm{e}^{-x_i/\mathrm{e}bs} \ge x_i/(\mathrm{e}^2bs),$$

and therefore

$$\mathbb{P}(X_s \le \mathrm{e}bs) \le \exp\left(1 - \frac{a}{be^3}\sum_{i\in\mathcal{I}} x_i^2\right).$$

Applying successively this inequality for $s = \mathrm{e}^j t_0$ and $j = 0, \dots, \lfloor \ln(t_1/t_0) \rfloor$, we conclude that

$$\mathbb{P}\left(X_{\mathrm{e}^jt_0} \le b\mathrm{e}^{j+1}t_0 \text{ for some } j = 0, \dots, \lfloor \ln(t_1/t_0)\rfloor\right) \le (1+\ln(t_1/t_0))\exp\left(1 - \frac{a}{be^3}\sum_{i\in\mathcal{I}} x_i^2\right),$$

which yields our claim by an argument of monotonicity. □

We next deduce from Lemma 9 and Proposition 4 an explicit lower-bound for the first saturation time T_n.

Corollary 3 *Suppose $\alpha < 2/3$ and pick $\beta \in \left(\frac{2}{3} \vee (1-2\alpha/3), 1-\alpha/2\right)$. Set $\ell_n = \lfloor n^{2\beta+\alpha-1}\rfloor$. Then*

$$\lim_{n\to\infty} \mathbb{P}(T_n \le \Gamma_{n,n+\ell_n}) = 0.$$

Proof: Let $k_n = n - m_n = \lfloor n^\beta \rfloor$ and note that $k_n \le \ell_n$. We shall implicitly work on the event

$$j/2n \le \Gamma_{n,m_n+j} - \Gamma_{n,m_n} \le 2j/n \qquad \text{for all } j \ge k_n - 1$$

as, by the laws of large numbers, the probability of this event is high when $n \to \infty$.

Our approach can be described as follows. For every $s \ge 0$, let B_s denote the number of cars which have been burnt during the time interval $[\Gamma_{n,m_n}, \Gamma_{n,m_n} + s]$. We aim at showing that

$$B_s \ge 2ns \qquad \text{for all } k_n/2n \le s \le 2(k_n+\ell_n)/n$$

with high probability. Note that on this event, the total number of cars that are burnt between times Γ_{n,m_n} and Γ_{n,m_n+j} exceeds j for all $k_n \le j \le k_n + \ell_n$. Since $T_n > \Gamma_{n,n-1}$, the saturation of the parking cannot occur before the arrival of the $(n+\ell_n)$-th car on the preceding event.

Observe that we may work with the dynamics where the car arrival process is stopped after

Γ_{n,m_n}. Indeed, thanks to Lemma 4, it suffices to establish that the event

$$\Lambda_n'' = \{B_s'' \geq 2ns \qquad \text{for all } k_n/2n \leq s \leq 2(k_n + \ell_n)/n\}$$

has a high probability, where B_s'' stands for the number of cars which have been burnt during the time interval $[\Gamma_{n,m_n}, \Gamma_{n,m_n} + s]$ in these dynamics.

This enables us to apply Lemma 9. More precisely, we consider the occupation of the parking at time Γ_{n,m_n} and write x_i for the size of the i-th largest occupied interval. We take $a = n^{-\alpha}$ so that the rate ax_i at which x_i is marked corresponds to the rate at which a Molotov cocktail is thrown on the interval with size x_i, and then $B_s'' = X_s$. We also take $b = 2n$, $t_0 = k_n/2n$ and $t_1 = 2(k_n + \ell_n)/n$. Note that

$$(bt_0) \wedge 1/(at_1) \leq n^\beta \wedge (n^{2-2\beta}/2) = n^{2-2\beta}/2\,;$$

we get from an application of Lemma 9 that the conditional probability of the complementary event $(\Lambda_n'')^c$ given the x_i can be bounded from above by

$$c \ln n \times \exp\left(-\frac{1}{2e^3 n^{1+\alpha}} \sum_{i \in \mathcal{I}} x_i^2\right)$$

where $\mathcal{I} = \left\{i : x_i \leq n^{2-2\beta}/2\right\}$.

In the notation of Proposition 4, we have

$$\sum_{i \in \mathcal{I}} x_i^2 = n^3 k_n^{-2} \langle \mu_{n,m_n}, f \rangle$$

with $f(x) = x \mathbf{1}_{x<1/2}$, and it follows from Proposition 4 and the porte-manteau Theorem that $\langle \mu_{n,m_n}, f \rangle$ converges in probability to some strictly positive constant. As $3 - 2\beta > 1 + \alpha$, we conclude from Fatou's lemma that the probability of $(\Lambda_n'')^c$ tends to 0 as $n \to \infty$, which completes the proof. $\square$

We are now able to proceed to the proof of Theorem 1(iv).

Proof of Theorem 1(iv): We keep the notation of Corollary 3 and pick

$$\beta' \in \left(\frac{2}{3} \vee (1 - 2\alpha/3), \beta\right) \quad \text{and} \quad \eta \in (2 - \alpha - 2\beta, 2 - \alpha - 2\beta')\,.$$

We set $m_n' = n - \lfloor n^{\beta'} \rfloor$. We have shown in the proof of Theorem 1(iii) that the mean density at time $\Gamma_{n,m_n'} + n^{-\eta}$ is small with high probability. As less than $2n^{1-\eta}$ cars have arrived between $\Gamma_{n,m_n'}$ and $\Gamma_{n,m_n'} + n^{-\eta}$ with high probability, we have $\Gamma_{n,m_n'} + n^{-\eta} \leq \Gamma_{n,m_n + 2n^{1-\eta}}$.

On the other hand, $1-\eta < 2\beta+\alpha-1$, and thus $m_n + 2n^{1-\eta} \le n+\ell_n$ where $\ell_n = \lfloor n^{2\beta+\alpha-1} \rfloor$. According to Corollary 3, saturation does not occur before $\Gamma_{n,n+\ell_n}$ with high probability and since the mean intensity increases at most linearly with unit rate, saturation does not occur either before t for every $t<2$ with high probability $\square$

5 Proof of Proposition 4

We still need to establish Proposition 4. This will be achieved by showing first that certain events have a high probability. Let $(m_n, n \in \mathbb{N})$ be a sequence of integers with

$$n^{\frac{2}{3}\vee(1-2\alpha/3)} \ln^{4/3} n \ll n - m_n \ll n\,. \tag{9}$$

We introduce two other sequences $(j_n, n\in\mathbb{N})$ and $(\ell_n, n\in\mathbb{N})$ such that

$$j_n \ll n - m_n \ll \ell_n\,. \tag{10}$$

We will also require these sequences to fulfill certain conditions that will be specified later on. Recall that $G'_{n,m}$ and $D'_{n,m}$ denote the first vacant sites at the left and at the right of 0 just after the arrival of the m-th car for the dynamics without fires.

Consider the events

$$\Lambda_{n,1} = \left\{ G'_{n,m_n} > -n^2/j_n^2 \quad \text{and} \quad D'_{n,m_n} < n^2/j_n^2 \right\}\,,$$

$\Lambda_{n,2}$ that no Molotov cocktails are thrown on the arc $]-n^2/j_n^2, n^2/j_n^2]$ between the arrivals of the $(n-\ell_n)$-th and the m_n-th cars, i.e.

$$\Lambda_{n,2} = \left\{ M_t \notin]-n^2/j_n^2, n^2/j_n^2] \text{ for all } \Gamma_{n,n-\ell_n} \le t \le \Gamma_{n,m_n} \right\}\,.$$

Finally, set

$$b_n = \frac{n}{(n-m_n)\ln n}\,,$$

and consider the event $\Lambda_{n,3}$ that the total number of cars that have arrived on the arc $]-n^2/j_n^2, n^2/j_n^2]$ and have been burnt before the arrival of the $(n-\ell_n)$-th car is smaller than b_n, i.e.

$$\Lambda_{n,3} = \left\{ \sum_{i=-n^2/j_n^2+1}^{n^2/j_n^2} \delta_{n,n-\ell_n}(i) \le b_n \right\}\,.$$

Lemma 10 (i) *On the event $\Lambda_{n,1}\cap\Lambda_{n,2}\cap\Lambda_{n,3}$, the total number of cars that have arrived on* *the arc* $]G'_{n,m_n}, D'_{n,m_n}[$ *and have been burnt before the arrival of the* m_n*-th car is smaller than* b_n*, i.e.*

$$\sum_{G'_{n,m_n} < i < D'_{n,m_n}} \delta_{n,m_n}(i) \le b_n \,. \tag{11}$$

(ii) *We can chose the sequences* $(j_n, n \in \mathbb{N})$ *and* $(\ell_n, n \in \mathbb{N})$ *in such a way that*

$$\lim_{n\to\infty} \mathbb{P}(\Lambda_{n,1} \cap \Lambda_{n,2} \cap \Lambda_{n,3}) = 1\,.$$

As a consequence, provided that (9) *holds,* (11) *occurs with high probability*

Proof: (i) On the event $\Lambda_{n,1} \cap \Lambda_{n,2} \cap \Lambda_{n,3}$, no car is burnt on $]G'_{n,m_n}, D'_{n,m_n}[$ between the arrivals of the $(n-\ell_n)$-th and the m_n-th car. Note that the sites G'_{n,m_n} and D'_{n,m_n} are vacant for the dynamics without fires at least until time Γ_{n,m_n} and thus prevent the propagation of fires started outside $]G'_{n,m_n}, D'_{n,m_n}[$ to $]G'_{n,m_n}, D'_{n,m_n}[$ until that time. Thus the total number of cars that have arrived on $]G'_{n,m_n}, D'_{n,m_n}[$ and have been burnt before $\Gamma_{n,m}$ is bounded from above by the number of cars that have arrived on the arc $]-n^2/j_n^2, n^2/j_n^2]$ and have been burnt before the arrival of the $(n-\ell_n)$-th car, which in turn is at most b_n.

(ii) We shall write as usual $k_n = n - m_n$. First, as $j_n \ll k_n$, we know from Proposition 3 that $\mathbb{P}(\Lambda_{n,1})$ that can be made as close to 1 as we wish by choosing n sufficiently large. Further, we get by conditioning on $\Gamma_{n,n-\ell_n}$ and Γ_{n,m_n} that

$$\begin{aligned}
\mathbb{P}(\Lambda_{n,2}) &= \mathbb{E}\left(\exp(-2n^2 j_n^{-2} n^{-\alpha}(\Gamma_{n,m_n} - \Gamma_{n,n-\ell_n}))\right) \\
&= \left(\frac{n}{n + 2n^{2-\alpha} j_n^{-2}}\right)^{\ell_n - k_n} \\
&= \left(1 - \frac{2n^{2-\alpha} j_n^{-2}}{n + 2n^{2-\alpha} j_n^{-2}}\right)^{\ell_n - k_n} ,
\end{aligned}$$

where the second equality stems from the stationarity of the increments of Gamma processes. As $\ell_n \gg k_n$, if we further impose $n^{2-\alpha} j_n^{-2} \ll n$, then we get

$$\ln \mathbb{P}(\Lambda_{n,2}) \sim -2\ell_n n^{1-\alpha} j_n^{-2}\,.$$

Thus $\mathbb{P}(\Lambda_{n,2})$ is as close to 1 as we wish provided that n is large enough and

$$n^{(1-\alpha)/2} \ll j_n \quad \text{and} \quad \ell_n \ll j_n^2 n^{\alpha-1}\,. \tag{12}$$

Last, by Lemma 8 and Markov inequality, the probability of $\Lambda_{n,3}$ is at least

$$1 - 2cn^2 j_n^{-2} \frac{n^{1-\alpha}}{\ell_n b_n} = 1 - 2c \frac{n^{2-\alpha} k_n \ln n}{j_n^2 \ell_n} .$$

This is close to 1 when n is large whenever

$$\ell_n \gg n^{2-\alpha} j_n^{-2} k_n \ln n . \tag{13}$$

Recapitulating, the proof will be completed if we check that the requirements (10), (12) and (13) can be fulfilled simultaneously. We can take for instance

$$j_n = n^{(3-2\alpha)/4} k_n^{1/4} \ln n \quad \text{and} \quad \ell_n = \sqrt{n k_n} \ln n .$$

Indeed, we then have

$$n^{(1-\alpha)/2} \ll n^{(3-2\alpha)/4} \ll n^{(3-2\alpha)/4} k_n^{1/4} \ln n = j_n .$$

Next, from our assumption, $n^{1-2\alpha/3} \ln^{4/3} n \ll k_n$; raising this inequality to the cube yields

$$j_n^4 = n^{3-2\alpha} k_n \ln^4 n \ll k_n^4 ,$$

and hence $j_n \ll k_n$. As clearly $\ell_n \gg k_n$, we have checked (10). Then we also have

$$\ell_n = \sqrt{n k_n} \ln n \ll n^{\alpha-1} n^{3/2-\alpha} k_n^{1/2} \ln^2 n = n^{\alpha-1} j_n^2 ,$$

so (12) holds. Finally we observe that

$$\ell_n = \sqrt{n k_n} \ln n = \frac{n^{2-\alpha} k_n \ln n}{n^{3/2-\alpha} k_n^{1/2} \ln^2 n} \ln^2 n \gg j_n^{-2} n^{2-\alpha} k_n \ln n ,$$

which shows that (13) is fulfilled. $\square$

Lemma 10 is the key for a useful asymptotic lower bound for the distribution of the occupied interval containing a typical site. Specifically, denote by $G_{n,m}$ and $D_{n,m}$ the first vacant sites at the left and at the right of 0 just after the arrival of the m-th car, and recall that $G'_{n,m}$ and $D'_{n,m}$ denote the same quantities for the dynamics without fires. Introduce also the probability measure on $(-\infty,] \times [0,\infty)$

$$\nu(\mathrm{d}x, \mathrm{d}y) = \frac{1}{\sqrt{2\pi(y-x)^3}} \exp(-(y-x)/2) \mathrm{d}x \mathrm{d}y , \qquad x < 0 < y ,$$

and recall from Proposition 3 that ν is the limiting distribution of

$$\left(\frac{(n-m)^2}{n^2}\,G'_{n,m},\,\frac{(n-m)^2}{n^2}\,D'_{n,m}\right)$$

in the regime $n^{2/3} \ll n-m \ll n$.

Lemma 11 *For every $x < 0 < y$, we have*

$$\liminf \mathbb{P}\left(\frac{(n-m)^2}{n^2}\,G_{n,m} < x,\,\frac{(n-m)^2}{n^2}\,D_{n,m} > y\right) \geq \nu((-\infty,x)\times(y,\infty))\,.$$

in the regime $n, m \to \infty$ with

$$n^{\frac{2}{3}\vee(1-2\alpha/3)}\ln^{4/3} n \ll n-m \ll n\,.$$

Proof: We use the same notation as for Lemma 10. Denote by $I'_n =]G'_{n,m_n}, D'_{n,m_n}[$ the occupied interval containing 0 after the arrival of the m_n-th car in the dynamics without fires. According to Lemma 3, the event that $G_{n,m_n} < xn^2/k_n^2$ and $D_{n,m} > yn^2/k_n^2$ occurs whenever $[xn^2/k_n^2, yn^2/k_n^2] \subseteq I'_n$ and the total number of cars that have arrived on I'_n and have been burnt before $\Gamma_{n,m}$ is strictly less than the minimum of $R'_{n,m_n} = S'_{n,m_n} - \underline{S}'_{n,m_n}$ on $I'_{n,m}$. We know from Lemma 10 that on the event with high probability $\Lambda_{n,1}\cap\Lambda'_{n,1}\cap\Lambda''_{n,1}$, the number of such cars is bounded by $b_n = n/(k_n \ln n)$.

Now recall Proposition 3 and the notations (5) and (6). By the Skohorod representation theorem, we may assume that the convergence stated there holds almost surely. Fix $\varepsilon > 0$ arbitrarily small and observe that conditionally on $G < x-\varepsilon$ and $D > y+\varepsilon$, we have $\min_{x\leq u\leq y} X_u > 0$ a.s. Recall that $b_n \ll n/k_n$ and $n^{-1}k_n X'_{n,m_n}(\lfloor n^2 k_n^{-2} u\rfloor)$ converges to X_u uniformly on $u \in \mathbb{R}$. We deduce that the conditional probability of $\min_{j\in\mathcal{I}'_n} X'_{n,m_n}(j) > b_n$ given that $[(x-\varepsilon)n^2k_n^{-2}, (y+\varepsilon)n^2k_n^{-2}] \subset I'_n$ is as close to 1 as we wish whenever n is sufficiently large. Hence, thanks to Lemma 3,

$$\liminf \mathbb{P}\left(\frac{(n-m)^2}{n^2}\,G_{n,m} < x,\,\frac{(n-m)^2}{n^2}\,D_{n,m} > y\right)$$

can be bounded from below by

$$\liminf \mathbb{P}\left(\frac{(n-m)^2}{n^2}\,G'_{n,m} < x-\varepsilon,\,\frac{(n-m)^2}{n^2}\,D'_{n,m} > y+\varepsilon, \Lambda_{n,1}\cap\Lambda'_{n,1}\cap\Lambda''_{n,1}\right)\,.$$

By Lemma 10, the latter is identical to

$$\liminf \mathbb{P}\left(\frac{(n-m)^2}{n^2}\, G'_{n,m} < x-\varepsilon, \frac{(n-m)^2}{n^2}\, D'_{n,m} > y+\varepsilon\right)\,,$$

and we know from Proposition 3 that this quantity is given by $\nu((-\infty, x-\varepsilon)\times(y+\varepsilon,\infty))$. As ε can be chosen as small as we wish, this establishes our claim. □

We may now proceed with the proof of Proposition 4.

Proof of Proposition 4: By the same argument of propagation of chaos as in the proof of Corollary 2, we just need to establish that after rescaling by the factor $(n-m)^2n^{-2}$, the length $L_{n,m}$ of the occupied interval which contains the site 0 converges in distribution to μ. We known from Lemma 2 that $D_{n,m} \le D'_{n,m}$ and $G'_{n,m} \le G_{n,m}$, and also from Proposition 3 that the distribution of

$$\left(\frac{(n-m)^2}{n^2}\, G'_{n,m}, \frac{(n-m)^2}{n^2}\, D'_{n,m}\right)$$

converges weakly in the regime $n^{2/3} \ll n-m \ll n$ towards

$$\nu(\mathrm{d}x,\mathrm{d}y) = \frac{1}{\sqrt{2\pi(y-x)^3}}\exp(-(y-x)/2)\mathrm{d}x\mathrm{d}y\,, \qquad x<0<y\,.$$

This ensures that for every $x<0<y$,

$$\limsup \mathbb{P}\left(\frac{(n-m)^2}{n^2}\, G_{n,m} < x, \frac{(n-m)^2}{n^2}\, D_{n,m} > y\right) \le \nu((-\infty,x)\times(y,\infty))\,.$$

The converse lower bound for the $\liminf$ has been established in Lemma 11, which completes the proof of our claim. □

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